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Path-Dependent Phase Shifts in Isolated Systems

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The Aharonov–Casher effect in a closed system is discussed. In this model, the charge on the wire is produced by a conducting bar moving in a magnetic field. If one considers the neutron to be a classical particle and the moving bar to be a quantum object, then the wave function of the bar acquires a phase shift equal in magnitude but opposite in sign to the usual phase shift of the neutron wave function. It is also shown that in any closed system, a path-dependent phase shift of one part of the system is always accompanied by an opposite phase shift of the remainder of the system. This result follows directly from the principle of least action.

1. INTRODUCTION

In a recent paper by Henneberger and Opatrný (1994), it was shown that the external field approximation can lead to wave functions in which single-valuedness is forfeited because of the approximation made. The Aharonov–Bohm (AB) effect was cited as an example. Specifically, it was shown that when the solenoid of the AB effect is represented as a spinning charged cylinder, the spinning cylinder undergoes a phase shift equal in magnitude but opposite in sign to that of the electron.

The thoughtful reader might conclude that this cancellation of phase shifts is no mere accident. In the present work, it is shown that such a cancellation always occurs in isolated systems in which the parts of a very weakly interacting system undergo path-dependent phase shifts. We first present the Aharonov–Casher (AC) effect in an isolated system as an additional example. In conclusion, a general theorem dealing with path-dependent phase factors in isolated systems is presented.

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2. THE AHARONOV–CASHER EFFECT

Aharonov and Casher (1984) showed that a magnetic dipole (a neutron) aligned parallel to a static line charge (a charged wire) will experience no force; however, paths in the plane perpendicular to the line charge passing it on opposite sides give a relative phase shift. Experimental verification of the effect was obtained by Kaiser *et al.* (1988) and by Cimmino *et al.* (1989).

The AC effect is a consequence of the fact that a moving magnetic moment gives rise to an electric dipole moment

$$\mathbf{p} = \frac{1}{c}\mathbf{v} \times \boldsymbol{\mu} \tag{1}$$

The energy of interaction with the line charge is then

$$U = -\mathbf{p}\cdot\mathbf{E} = \frac{1}{c}\mathbf{v}\cdot\mathbf{E} \times \boldsymbol{\mu} \tag{2}$$

where $\mathbf{E}$ is the electric field of the line charge.

The interaction of equation (2) has a long history in physics. It was shown long ago by Thomas (1926) and Frenkel (1926) that this interaction is responsible for spin–orbit coupling. A complete nonrelativistic derivation of spin–orbit coupling based on this interaction was recently given by Al-Jaber *et al.* (1991). These authors show the spin–orbit interaction to be due to Larmor precession of the orbit due to an effective magnetic field given by

$$\mathbf{B}_{\text{eff}} = -\frac{1}{e}(\boldsymbol{\mu}\cdot\nabla)\mathbf{E} \tag{3}$$

Still more recently, the present authors have argued that the energy correction for *s*-states first obtained by Darwin (1928) is due to a quadratic term in the effective vector potential

$$\mathbf{A}_{\text{eff}} = -\frac{1}{e}(\mathbf{E} \times \boldsymbol{\mu}) \tag{4}$$

in the expression for the electron's kinetic energy.

As shown by Henneberger and Opatrný (HO) (1994), if one insists upon single-valued wave functions in problems dealing with path-dependent phase shifts, one must consider an isolated system. We call the reader's attention to the fact that the usual discussions of the AB and AC effects involve an exchange of energy of the solenoid (in AB) or the charged wire (in AC) with the environment. The purpose of the HO paper was to make this energy exchange clear, and to show that the angular position of the rotating cylinder slightly leads or lags the position it would have, had the electron not passed.

In a discussion of the AC effect, it is likewise necessary to consider a closed system. The configuration of our *Gedanken* experiment is given in Fig. 1. A moving bar slides without friction over conducting rails separated by a distance D. The space between the rails has a magnetic field $\mathbf{B}$ directed upward. The rails are very long and there is no resistance anywhere. At $t = -\infty$, the rod of mass M moves to the right with speed V. One rail is grounded; the other is connected to the vertical wire that supplies the electric field for the AC effect. At $t = 0$, the neutron (assumed to have its magnetic moment up) passes the point of closest approach to the wire. The voltage generated in the moving bar supplies the charge to the wire (which is extremely long). The wire has capacitance per centimeter C. In this computation, in order to demonstrate the phase shift of the rod, we treat the neutron as a classical particle. Since the neutron experiences no force, this is a valid approximation.

We begin by noting the topological phase factor given by Aharonov and Casher (1984). It is $\exp[i\Phi_{AC}(\mathbf{r})]$, with

$$\Phi_{AC}(\mathbf{r}) = \frac{1}{c\hbar}\int_{-\infty}^{r} \boldsymbol{\mu} \times \mathbf{E}\cdot d\mathbf{r} \tag{5}$$

In Fig. 1, the wire runs along the z axis.

The azimuthal angle φ is given by

$$\sin\varphi = \frac{x}{(a^2 + x^2)^{1/2}} \tag{6}$$

where a is the distance of closest approach to the wire (impact parameter) of the neutron. The electric field is then given by

$$\mathbf{E} = \hat{i}\,\frac{2\lambda}{(a^2 + x^2)^{1/2}}\sin\varphi - \hat{j}\,\frac{2\lambda\cos\varphi}{(a^2 + x^2)^{1/2}} \tag{7}$$

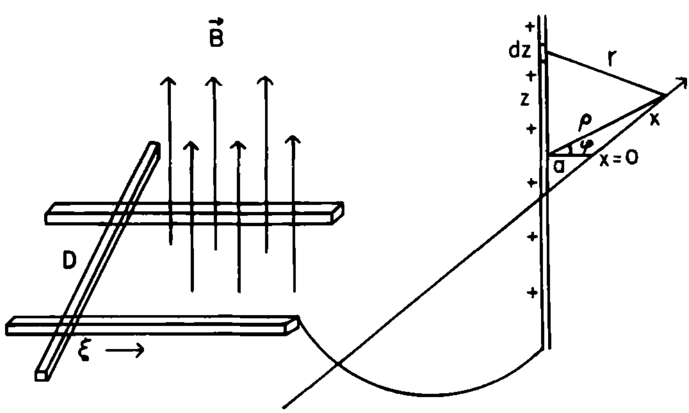

Fig. 1.

where λ is the charge/length on the wire. This yields

$$\Phi_{AC}(x) = \int_{-\infty}^{x} \frac{\mu}{c\hbar} \frac{2\lambda a}{(a^2 + x^2)} \, dx = \frac{2\mu\lambda}{c\hbar} \left(\varphi + \frac{\pi}{2} \right) \tag{8}$$

The interaction of equation (2) in the situation of Fig. 1 is

$$U = -\frac{2\lambda v \mu a}{c(x^2 + a^2)} \tag{9}$$

with $x = vt$.

It is straightforward to relate the motion of the neutron to the Lagrangian of the moving bar. The charge/cm on the wire is related to the speed of the moving bar by

$$\lambda = \frac{C}{c} BD\dot{\xi} \tag{10}$$

where $\dot{\xi}$ is the velocity of the moving bar and D is the distance between the parallel rails. The interaction energy of equations (2) and (9) now becomes

$$U = -\frac{2CBDv\mu a\dot{\xi}}{c^2(x^2 + a^2)} \tag{11}$$

The Lagrangian for the moving bar is thus

$$L = \frac{1}{2} M\dot{\xi}^2 + \frac{2CBDv\mu a\dot{\xi}}{c^2(x^2 + a^2)} \tag{12}$$

The canonical momentum conjugate to ξ is

$$p_\xi = \frac{\partial L}{\partial \dot{\xi}} = M\dot{\xi} + \frac{2CBDv\mu a}{c^2(x^2 + a^2)} \tag{13}$$

Since $\partial L/\partial \xi = 0$, p_ξ is a conserved quantity.

The value of this integral of motion is clearly MV, where V is the velocity of the bar at $t = -\infty$. The value of $\dot{\xi}$ at time t is therefore given by

$$\dot{\xi} = V - \frac{2CBDv\mu a}{Mc^2(x^2 + a^2)} = V + \Delta\dot{\xi} \tag{14}$$

The phase shift of the moving bar is then given by

$$\Delta\Phi_{\text{bar}}(x) = \frac{\Delta \int p_\xi \, d\xi}{\hbar} = \frac{1}{\hbar} \int p_\xi \, \Delta\dot{\xi} \, dt \tag{15}$$

where $dt = dx/v$. This yields

$$\Delta\Phi_{\text{bar}}(x) = -\frac{MV}{\hbar}\frac{2CBD\mu a}{Mc^2}\int_{-\infty}^{x}\frac{dx}{x^2 + a^2} \tag{16}$$

with $\lambda = (C/c)BDV$; this is just the negative of the AC phase shift of equation (8).

3. ENERGY CONSERVATION

The Lagrangian formulation of the problem leads to results in such a direct manner that the physics of a problem is often obscured. It is therefore profitable to consider energy conservation in the system discussed here.

We begin with the observation that the additional electric field energy $(1/4\pi)\int \mathbf{E}_{\text{neut}}(\mathbf{r}' - \mathbf{r})\cdot\mathbf{E}_{\text{wire}}(\mathbf{r}')\,d^3\mathbf{r}'$ due to the passing neutron is zero. The force on the neutron is the negative gradient of this energy. However, the force on the neutron is zero. Therefore, this overlap energy is constant. Evaluation of this constant energy at a time when the neutron is infinitely far from the wire shows that the overlap field energy vanishes.

We next assume that the passage of the neutron is adiabatic, i.e., that the wire is always in a state of static equilibrium. The surface of the wire is therefore an equipotential surface at all times. A slight redistribution of the charge on the wire must give a potential that just cancels the potential of the passing neutron. We thus have

$$\Delta\lambda(z) = -C\Delta\phi(z) \tag{17}$$

where $\Delta\phi(z)$ is the potential of the passing neutron at point z in the wire. The change in energy of the wire is

$$\int_{-\infty}^{\infty} \epsilon\,\Delta\lambda(z)\,dz = \frac{1}{c}BDV\int_{-\infty}^{\infty}\Delta\lambda(z)\,dz = -\frac{CBDV}{c}\int_{-\infty}^{\infty}\Delta\phi(z)\,dz \tag{18}$$

where ϵ is the voltage induced on the wire by the moving bar. The potential at a point z on the wire is

$$\Delta\phi(\mathbf{r}) = \frac{\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{19}$$

where $\mathbf{r} = \hat{k}z$ and $\mathbf{r}' = \hat{i}x - \hat{j}a$. Then $\mathbf{p} = (1/c)\mathbf{v} \times \boldsymbol{\mu} = -\hat{j}\,v\mu/c$ yields

$$\Delta\phi(z, x) = -\frac{v\mu a}{c}\frac{1}{(x^2 + a^2 + z^2)^{3/2}} \tag{20}$$

Equations (18) and (20) yield a change in the potential energy of the wire given by

$$\Delta U_{\text{wire}} = \frac{CBDVv\mu a}{c^2} \int_{-\infty}^{\infty} \frac{dz}{(x^2 + a^2 + z^2)^{3/2}} = \frac{2CBDV}{c^2} \frac{v\mu a}{(x^2 + a^2)} \tag{21}$$

The change in the kinetic energy of the moving bar is

$$\Delta\left(\frac{1}{2} M\dot{\xi}^2\right) = M\dot{\xi}\, \Delta\dot{\xi} \approx MV\, \Delta\dot{\xi} \tag{22}$$

Then $\Delta\dot{\xi}$ of equation (14) gives

$$\Delta\left(\frac{1}{2} M\dot{\xi}^2\right) = \frac{-2VCBDv\mu a}{c^2(x^2 + a^2)} \tag{23}$$

This is the negative of the change in potential energy of the wire. The reader who remains skeptical may wish to check that the force on the bar given by

$$F_\xi = -\frac{BDi}{c} \tag{24}$$

with

$$i = \frac{d}{dt} \int \Delta\lambda(z)\, dz \tag{25}$$

is equal to $M\ddot{\xi}$ with

$$\ddot{\xi} = \frac{d}{dt} (\Delta\dot{\xi})$$

4. A GENERAL THEOREM

In this paper and in an earlier one (HO), examples have been given which demonstrate that in an isolated system, when one part of the system experiences a path-dependent phase shift because of a very small interaction with the remaining parts of the system, the remaining parts undergo an equal but opposite phase shift. This is certainly no accident. In the following, we demonstrate that this phenomenon is a necessary consequence of the principle of least action.

We consider an isolated system having N degrees of freedom with coordinates $q_1, q_2, \ldots, q_N$. The system is described by a Lagrangian L, so that the canonically conjugate momenta are as usual given by

$$p_k = \frac{\partial L}{\partial \dot{q}_k}, \qquad k = 1, \ldots, N \tag{26}$$

The principle of least action then states that

$$\delta \int_{t_1}^{t_2} L\, dt = 0, \qquad \text{with} \quad L = \sum_{k=1}^{N} p_k \dot{q}_k - H \tag{27}$$

where the second of the equations is the defining equation for H. The above equations yield

$$\delta \int_{t_1}^{t_2} \sum_{k=1}^{N} p_k \dot{q}_k \, dt - \delta \int_{t_1}^{t_2} H\, dt = 0 \tag{28}$$

The above equations hold for arbitrary infinitesimal variation in coordinates and momenta. Instead of considering arbitrary variations, we consider first the system without the interaction. Interactions that result only in a path-dependent phase factor are, by their nature, extremely weak. We thus consider the variations δq_k and δp_k to be those resulting in an adiabatic switching on and off of the interaction leading to the phase shifts. We consider free particle effects such as those of AB and AC.

Now, H is conserved, and the adiabatic switching on and off of the interaction occurs when the interacting parts are separated by very great distances. Hence we have

$$\delta \int_{t_1}^{t_2} H\, dt = 0 \tag{29}$$

Thus, equation (28) tells us that

$$\delta \int_{t_1}^{t_2} \sum_{k=1}^{N} p_k \dot{q}_k \, dt = \delta \sum_{k=1}^{N} \int_{q_k(t_1)}^{q_k(t_2)} p_k \, dq_k = 0 \tag{30}$$

But

$$\delta \int_{q_k(t_1)}^{q_k(t_2)} p_k \, dq_k$$

is just the phase shift associated with the coordinate q_k, in units of $\hbar$.

We note that here the δq_k do not necessarily vanish at the endpoints of the integral. The variations here are not virtual displacements, but real ones. This is quite legitimate. In the usual theory, the δq_k are chosen to vanish at the endpoints of the time integral for convenience, since the displacements are only virtual. All that is really required is the variational condition (27).

We see that $(1/\hbar) \sum_{k=1}^{N} \delta \int p_k \, dq_k = 0$ for any isolated weakly interacting system. This result appears almost obvious, yet its implications are quite profound. It means that one must use utmost care in splitting an interacting system into a particle and an external field. We see that whenever the wave function of a particle in such a system undergoes a phase shift, the wave function of the rest of the system undergoes an opposite phase shift. To ignore this phase shift of the remainder of the system is to violate the postulate of quantum theory that requires the wave function of the completely isolated system to be single-valued. The external field approximation is just that—an approximation. This has been discussed in detail in HO.

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